

Problem 1

Q1

You roll 15 fair dice. What is the probability that none of them shows a 6?

Solution. Each die independently shows a non-six with probability $\frac{5}{6}$.

By independence,

$$P = \left(\frac{5}{6}\right)^{15} = \frac{5^{15}}{6^{15}} = \frac{30517578125}{470184984576} \approx 0.064906$$

Result: 0.0649

Q2

Four coins: three fair coins and one two-headed coin.
You pick one at random (without examining it) and toss it twice.
What is the probability of two heads?

Solution. Condition on which coin was drawn (law of total probability):

$$P(HH) = P(\text{fair}) P(HH \mid \text{fair}) + P(\text{2-headed}) P(HH \mid \text{2-headed}) = \frac{3}{4} \times \frac{1}{4} + \frac{1}{4} \times 1 = \frac{3}{16} + \frac{4}{16} = \frac{7}{16} \approx 0.4375$$

Result: 0.4375

Q3

Given that both tosses came up heads, what is the probability the coin was the two-headed one?

Solution. By Bayes' theorem, using $P(HH) = \frac{7}{16}$ from Q2:

$$P(2\text{-headed} \mid HH) = \frac{P(HH \mid 2\text{-headed}) P(2\text{-headed})}{P(HH)} = \frac{1 \times \frac{1}{4}}{\frac{7}{16}} = \frac{\frac{4}{16}}{\frac{7}{16}} = \frac{4}{7}$$

Result: 0.5714

Q4

Three points are dropped independently and uniformly on the segment $[0, 1]$. Find the probability that the third point falls between the first and the second.

Solution. The three coordinates X_1, X_2, X_3 are i.i.d. continuous, so all $3! = 6$ orderings are equally likely. "The third point lies between the other two" means X_3 is the median, i.e. the orderings $X_1 < X_3 < X_2$ or $X_2 < X_3 < X_1$ — exactly 2 of the 6 cases. Hence

$$P = \frac{2}{6} = \frac{1}{3} \approx 0.3333$$

Result: 0.3333