

Problem 3

Q1

In $\mathbb{R}^2$, vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ satisfy $|\vec{u}| = |\vec{v}| = \sqrt{5}$, $\vec{u} \circ \vec{v} = 3$ (acute), $|\vec{w}| = \sqrt{37}$, $\vec{u} \circ \vec{w} = 13$, and $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent. Find $\vec{v} \circ \vec{w}$.

Solution. Write the Gram matrix of pairwise dot products, with the unknown $X = \vec{v} \circ \vec{w}$:

$$G = \begin{pmatrix} \vec{u} \circ \vec{u} & \vec{u} \circ \vec{v} & \vec{u} \circ \vec{w} \\ \vec{v} \circ \vec{u} & \vec{v} \circ \vec{v} & \vec{v} \circ \vec{w} \\ \vec{w} \circ \vec{u} & \vec{w} \circ \vec{v} & \vec{w} \circ \vec{w} \end{pmatrix} = \begin{pmatrix} 5 & 3 & 13 \\ 3 & 5 & X \\ 13 & X & 37 \end{pmatrix}$$

Three vectors are linearly dependent $\iff \det G = 0$:

$$\det G = -5X^2 + 78X - 253 = 0 \Rightarrow 5X^2 - 78X + 253 = 0 \Rightarrow X = \frac{78 \pm 32}{6} = 11 \text{ or } 4.6$$

Both roots are geometrically realizable. The natural decomposition in the basis $\{\vec{u}, \vec{v}\}$ is $\vec{w} = 2\vec{u} + \vec{v}$ (which indeed gives $\vec{u} \circ \vec{w} = 2(5) + 3 = 13$ and $|\vec{w}|^2 = 4(5) + 5 + 4(3) = 37$), yielding

$$\vec{v} \circ \vec{w} = 2(\vec{u} \circ \vec{v}) + (\vec{v} \circ \vec{v}) = 2(3) + 5 = 11$$

Result: 11.0000 (the alternative configuration $\vec{w} = 3.2\vec{u} - \vec{v}$ gives 4.6000)

Q2

For $n \geq 1$, let $A_n = \int_0^1 (1 - x^2)x^{n-1} dx$. Compute $A_1 + A_2$

Solution. Split the integral:

$$A_n = \int_0^1 x^{n-1} dx - \int_0^1 x^{n+1} dx = \frac{1}{n} - \frac{1}{n+2}$$

Hence $A_1 = 1 - \frac{1}{3} = \frac{2}{3}$ and $A_2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$, so

$$A_1 + A_2 = \frac{2}{3} + \frac{1}{4} = \frac{8+3}{4} = \frac{11}{12} \approx 0.9167$$

Result: 0.9167

Q3

At what angle (in radians) do the curves $y = x^2$ and $x = y^2$ intersect at the upper-right intersection point?

Solution. Intersections satisfy $x = x^4 \Rightarrow x \in \{0, 1\}$, giving $(0, 0)$ and the upper-right point $(1, 1)$. The slopes there are

$$y = x^2: \frac{dy}{dx} = 2x \Big|_{x=1} = 2, \quad y = \sqrt{x}: \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \Big|_{x=1} = \frac{1}{2}$$

The angle between the curves is

$$\theta = \arctan \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = \arctan \left| \frac{2 - \frac{1}{2}}{1 + 2 \times \frac{1}{2}} \right| = \arctan \frac{3}{4} \approx 0.6435 \text{ rad}$$

Result: 0.6435